JOURNAL OF COMBINATORIAL THEORY (A) 23, 148–175 (1977)

Partition Identities and Invariants of Finite Groups*

LOUIS SOLOMON

University of Wisconsin, Madison, Wisconsin 53706

Communicated by the Managing Editors

Received October 27, 1976

TO THE MEMORY OF EDWARD KOBAYASHI

1. INTRODUCTION

Let m be a positive integer. Let K be the set of all m-tuples $\kappa = (k_1, \ldots, k_m)$ of nonnegative integers. Let $x_1, \ldots, x_m$ be indeterminates and write $x^\kappa = x_1^{k_1} \cdots x_m^{k_m}$. In this paper we interpret the coefficients Q_n and $\hat{Q}_n$ in certain formal products

$$\prod_{\kappa \in K} (1 - x^\kappa t)^{-e_\kappa} = \sum_{n=0}^{\infty} Q_n t^n, \tag{1.1}$$

$$\prod_{\kappa \in K} (1 + x^\kappa t)^{e_\kappa} = \sum_{n=0}^{\infty} \hat{Q}_n t^n \tag{1.2}$$

as Poincaré series which occur in the invariant theory of a finite group G of linear transformations.

The simplest cases of (1.1) and (1.2) are Euler's identities

$$\prod_{k=0}^{\infty} (1 - x^k t)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{(1-x) \cdots (1-x^n)} t^n, \tag{1.1$'$}$$

$$\prod_{k=0}^{\infty} (1 + x^k t) = 1 + \sum_{n=1}^{\infty} \frac{x^{n(n-1)/2}}{(1-x) \cdots (1-x^n)} t^n, \tag{1.2$'$}$$

which correspond to $m = 1$ and $G = 1$. The method of this paper tells us to prove (1.1′) and (1.2′) as follows. Interpret the coefficient of t^n in terms of symmetric or alternating functions of n indeterminates, apply Molien's formula to introduce a summation over the symmetric group S_n, and then rearrange the right-hand side.

* This work was supported in part by the National Science Foundation.

To give some idea of the main results of this paper and, at the same time, to avoid technical complication for the moment, we assume in this introduction that m is arbitrary and $e_\kappa = 1$ for all κ. This corresponds to the case $G = 1$. These series Q_n were studied by MacMahon [11, Sect. 301] because they are generating functions for the number of partitions of an m-partite number. They may also be interpreted in terms of the theory of symmetric functions of several sets of indeterminates [11, Sect. 535–538; 21, Sect. 24–25]. Andrews [1, Chap. 12] gives a good summary of their properties and a useful bibliography. The series P_n and $\hat{P}_n$ defined by

$$P_n = Q_n \prod_{i=1}^{m} (1 - x_i)(1 - x_i^2) \cdots (1 - x_i^n), \tag{1.3}$$

$$\hat{P}_n = \hat{Q}_n \prod_{i=1}^{m} (1 - x_i)(1 - x_i^2) \cdots (1 - x_i^n) \tag{1.4}$$

are in fact polynomials, and we shall see that these polynomials may also be interpreted as Poincaré series. The associated multigraded vector spaces may be constructed from the space H^n of polynomials harmonic for the symmetric group S_n [8, 20], those polynomials in n indeterminates $X_1, \ldots, X_n$ which lie in the kernels of all differential operators $f(\partial/\partial X_1, \ldots, \partial/\partial X_n)$ where f is a symmetric polynomial in $X_1, \ldots, X_n$ with $f(0, \ldots, 0) = 0$.

This interpretation of the P_n as Poincaré series explains much of what one knows about them. For example, Wright [24] conjectured and Gordon [9] proved by intricate combinatorial argument that the P_n have nonnegative integer coefficients. This is clear from our results since a Poincaré series has nonnegative coefficients by definition. Gordon's intricate combinatorics is replaced by Lemma 4.3 below, which lies at about the level of difficulty of the Macaulay unmixedness theorem, and transfers the difficulty from combinatorics to commutative algebra. One corollary of this lemma is Theorem 4.26, a generalization of Gordon's theorem which corresponds to a cyclic group G but involves no group invariants in its statement. In another direction Corollary 4.4 is a refinement of Burnside's theorem on tensor powers of a representation of a finite group.

Let λ be a partition of n and let χ be the corresponding character of S_n. Let H_χ^n be the isotypic component of H^n of type χ, by definition the sum of all simple submodules of H^n which have character χ. In Section 5 we compute the Poincaré series of H_χ^n and give a formula for P_n as a multiple sum involving the Poincaré series of H_χ^n and the irreducible characters of S_n. There is a formal similarity between the computation in Theorem 5.18 and some of Stanley's work [18] on plane partitions, but plane partitions do not enter into this paper.

In case $m = 2$ the formula for P_n simplifies because of the orthogonality relations for the characters, and one can use the formula to compute the polynomials P_n for small n. I did this about 10 years ago in connection with a problem on Weyl groups and Chevalley groups which had nothing to do with the infinite products (1.1) and (1.2). The results were not what one might have hoped for and the work languished until, by fortunate coincidence, I stumbled on a paper by Carlitz [6], who had computed precisely the same polynomials from a different point of view The source of the present paper thus lies in Example 5.21, which appears in [6].

In Section 6 we study the connection between products of the form (1.1) and (1.2), and the Polya–Redfield counting formula, a connection which Polya noticed in 1937 [13]. Formula (6.4) includes both the Polya–Redfield formula and Lemma 2.6. Nevertheless, in spite of the seeming generality of (6.4) it seems that one cannot get at any of the results of Sections 4 and 5 in this way because one cannot apply Lemma 3.4 in Polya's setup.

I would like to thank George Andrews for telling me about Gordon's proof of Wright's conjecture and for encouraging me to write this paper.

2. Poincaré Series and Wreath Products

Let G be a finite group and let V be a G module. We choose the field $\mathbf{C}$ of complex numbers as ground field as a matter of convenience, so that G module means $\mathbf{C}[G]$ module; with some minor rewriting we could replace $\mathbf{C}$ by any characteristic zero splitting field for G. If $g \in G$ we let $g \mid V$ denote the endomorphism of V defined by g. We say that V is a graded G module if $V = \bigoplus_k V_k$ is a graded vector space in which the homogeneous components V_k are G modules of finite dimension. Sums over k are understood to be over all nonnegative integers. If $g \in G$ we define a formal power series in an indeterminate x by

$$V(x, g) = \sum_k \operatorname{tr}(g \mid V_k)\, x^k, \tag{2.1}$$

where tr denotes the trace. If $g = 1$ we simply write

$$V(x) = \sum_k \dim(V_k)\, x^k \tag{2.2}$$

for the Poincaré series of V.

Let $\Omega_n = \{1,..., n\}$ and let S_n be the symmetric group on Ω_n. Let $G_n = G \wr S_n$ be the wreath product of G with S_n. Recall [10] that G_n consists of all pairs (f, σ) where f is a mapping of Ω_n into G and $\sigma \in S_n$. The multiplication in G_n is defined by

$$(f_1, \sigma_1)(f_2, \sigma_2) = (f, \sigma_1\sigma_2),$$

where

$$f(p) = f_1(p) f_2(\sigma_1^{-1} p), \qquad p \in \Omega_n .$$

The set of all elements $(f, 1)$ is a normal subgroup of G_n isomorphic to $G \times \cdots \times G$, n factors. The set of all elements $(1, \sigma)$, where 1 denotes the constant map with value equal to the identity of G, is a subgroup isomorphic to S_n. Note that $(f, 1)(1, \sigma) = (f, \sigma)$. We identity G_1 with G and agree that $G_0 = 1$. Let $V^n = T^n V$ be the nth tensor power of V. Make V^n a graded vector space by defining

$$V_k{}^n = \bigoplus_{k_1+\cdots+k_n=k} V_{k_1} \otimes \cdots \otimes V_{k_n} \tag{2.3}$$

and give it a G_n module structure with the action

$$(f, \sigma)(v_1 \otimes \cdots \otimes v_n) = f(1)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes f(n)\, v_{\sigma^{-1}(n)} \tag{2.4}$$

for all $v_1, \ldots, v_n \in V$. Thus the formal power series $V_n(x, \tau)$ is defined for all $\tau \in G_n$. In this section we compute $V_n(x, \tau)$ in terms of the series $V(x, g)$ with $g \in G$.

Let $K_1, \ldots, K_r$ be the conjugacy classes of G. Choose, once and for all, an element $g_i \in K_i$. Fix an element $\tau = (f, \sigma)$ of G_n. If $\Delta \subseteq \Omega_n$ is a σ orbit, of cardinality say d, then the d products

$$f(p)\, f(\sigma^{-1}p)\, f(\sigma^{-2}p) \cdots f(\sigma^{1-d}p), \qquad p \in \Delta$$

differ from one another by cyclic permutations of the factors and hence are conjugate in G. Let $K(\tau, \Delta)$ denote the conjugacy class of G containing all these products. For $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$ let a_{ij} be the number of orbits Δ of σ such that $K(\tau, \Delta) = K_i$ and $|\Delta| = j$, where $|\Delta|$ denotes the cardinality of Δ. The $r \times n$ matrix $a = (a_{ij})$ is called the type of τ. The integers a_{ij} satisfy the conditions

$$a_{ij} \geqslant 0, \qquad \sum_{i,j} j a_{ij} = n. \tag{2.5}$$

Let $\mathscr{A}_n$ be the set of all $r \times \infty$ matrices of integers satisfying these conditions. We view an $r \times n$ matrix as an $r \times \infty$ matrix by inserting columns of zeros, so that the type of τ is an element of $\mathscr{A}_n$. Every element of $\mathscr{A}_n$ occurs as the type of some element of G_n and two elements of G_n are conjugate if and only if they have the same type [10, Sect. 3.7].

2.6. LEMMA. *Let $\tau \in G_n$ have type (a_{ij}). Then*

$$V^n(x, \tau) = \prod_{i,j} V(x^j, g_i)^{a_{ij}}.$$

Proof. Argue by induction on n. Say $\tau = (f, \sigma)$. Suppose first that σ has just one orbit, and without loss of generality that $\sigma = (n \cdots 21)$. Choose a basis for V^n adapted to the direct sum decomposition (2.3). Since τ maps $V_{k_1} \otimes \cdots \otimes V_{k_n}$ into $V_{k_2} \otimes \cdots \otimes V_{k_n} \otimes V_{k_1}$ there is no contribution to the trace unless $k_1 = \cdots = k_n$. Thus

$$V^n(x, \tau) = \sum_k \operatorname{tr}(\tau \mid T^n V_k)\, x^{nk}.$$

Fix k, let $u_1, u_2, \ldots$ be a basis for V_k, and for simplicity write $f_i = f(i)$. Then using the basis for $T^n V_k$ consisting of all elements $u_{i_1} \otimes \cdots \otimes u_{i_n}$ we see that

$$\operatorname{tr}(\tau \mid T^n V_k) = \operatorname{tr}(f_1 \cdots f_n \mid V_k)$$

and thus

$$V^n(x, \tau) = \sum_k \operatorname{tr}(f_1 \cdots f_n \mid V_k)\, x^{nk} = V(x^n, f_1 \cdots f_n).$$

This proves the lemma in case σ has just one orbit.

Suppose now that $n = n' + n''$ where n', n'' are positive integers. Identify $G_{n'} \times G_{n''}$ with a subgroup of G_n and $V^{n'} \otimes V^{n''}$ with V^n. If $\tau' \in G_{n'}$ and $\tau'' \in G_{n''}$ then setting $\tau = \tau'\tau'' \in G_n$ we have $\tau \mid V^n = (\tau' \mid V^{n'}) \otimes (\tau'' \mid V^{n''})$. Since

$$V_k{}^n = \bigoplus_{k'+k''=k} V_{k'}^{n'} \otimes V_{k''}^{n''}$$

we have

$$\begin{aligned} \operatorname{tr}(\tau \mid V_k{}^n) &= \sum_{k'+k''=k} \operatorname{tr}((\tau' \mid V_{k'}^{n'}) \otimes (\tau'' \mid V_{k''}^{n''})) \\ &= \sum_{k'+k''=k} \operatorname{tr}(\tau' \mid V_{k'}^{n'}) \operatorname{tr}(\tau'' \mid V_{k''}^{n''}). \end{aligned}$$

Thus

$$V^n(x, \tau) = V^{n'}(x, \tau')\, V^{n''}(x, \tau''). \tag{2.7}$$

If τ' has type (a'_{ij}) and τ'' has type (a''_{ij}) then σ has type (a_{ij}) where $a_{ij} = a'_{ij} + a''_{ij}$. Thus using (2.7) and the induction hypothesis we get

$$V^n(x, \tau) = \prod_{i,j} V(x^j, g_i)^{a_{ij}}.$$

Since any element $\tau \in G_n$ for which the corresponding τ has more than one orbit may be decomposed as $\tau'\tau''$ for suitable τ', τ'' the proof is complete. $\square$

If $V_k = 0$ for positive k so that V is a finite-dimensional G module with no grading, then Lemma 2.6 amounts to formula (41) of Specht's thesis [17]. In this form it may be traced back to Schur. If f is constant, say $f(i) = g$ for all $i \in \Omega_n$, then writing $A = g \mid V$ (2.6) becomes

$$\operatorname{tr}(\sigma \cdot T^n A) = \operatorname{tr}(A)^{a_1} \operatorname{tr}(A^2)^{a_2} \cdots \operatorname{tr}(A^n)^{a_n},$$

where T^nA is the nth tensor power of A and a_j is the number of σ orbits with cardinality j. This is the formula which Schur [15, Hilfsatz 3] used to study the characters of S_n and $GL(V)$.

3. Poincaré Series and Infinite Products

We say that a vector space is multigraded if it is a direct sum of finite-dimensional spaces which are indexed by m-tuples $\kappa = (k_1, ..., k_m)$ of non-negative integers. If $V = \bigoplus_k V_k$ is a graded vector space, then the mth tensor power T^mV is multigraded by

$$T^mV = \bigoplus_{\kappa} (T^mV)_\kappa \tag{3.1}$$

where

$$(T^mV)_\kappa = V_{k_1} \otimes \cdots \otimes V_{k_m}. \tag{3.2}$$

Sums over κ are understood to be over the set K of all m-tuples of nonnegative integers. If V is a graded G module we give T^mV the G module structure

$$g(v_1 \otimes \cdots \otimes v_m) = gv_1 \otimes \cdots \otimes gv_m, \qquad g \in G,\ v_i \in V. \tag{3.3}$$

Let $x_1, ..., x_m$ be interdeterminates. Write $x = (x_1, ..., x_m)$[1] and $x^\kappa = x_1^{k_1} \cdots x_m^{k_m}$. If $g \in G$ we define a formal power series $(T^mV)(x, g)$ by

$$(T^mV)(x, g) = \sum_{\kappa} \operatorname{tr}(g \mid (T^mV)_\kappa)\, x^\kappa. \tag{3.4}$$

It follows from (3.3) that

$$(T^mV)(x, g) = \prod_{h=1}^{m} V(x_h, g). \tag{3.5}$$

If $g = 1$ we simply write

$$(T^mV)(x) = \sum_{\kappa} \dim((T^mV)_\kappa)\, x^\kappa \tag{3.6}$$

for the Poincaré series of the multigraded space T^mV.

Let n be a positive integer and let $V^n = T^nV$ be the n-fold tensor power of V. Give V^n the grading (2.3) and the G_n-module structure (2.4). We may apply the constructions of the preceding paragraph with G_n in place of G and V^n in place of V. Formula (3.5) is then replaced by

$$(T^mV^n)(x, \tau) = \prod_{h=1}^{m} V^n(x_h, \tau). \tag{3.7}$$

[1] There should be no confusion with the same letter x used as a single indeterminate in the Poincaré series of a graded vector space. This is a convenient abuse of notation but the reader should be careful with some of the formulas in Sections 4 and 5.

If τ has type (a_{ij}) it follows from (2.6) that

$$(T^m V^n)(x, \tau) = \prod_{h,i,j} V(x_h{}^j, g_i)^{a_{ij}}. \tag{3.8}$$

Let W be a G module of finite dimension. Then W affords a character, say ψ, of G. Let $W^n = T^n W$ and give W^n the G_n module structure (2.4). If we apply (2.6) to the space V defined by $V_0 = W$ and $V_k = 0$ for $k > 0$ we get

$$\operatorname{tr}(\tau \mid W^n) = \prod_{i,j} \psi(g_i)^{a_{ij}}. \tag{3.9}$$

Multigrade the space $T^m V^n \otimes W^n$ by

$$(T^m V^n \otimes W^n)_\kappa = (T^m V^n)_\kappa \otimes W^n$$

and give it the G_n module structure

$$\tau(v \otimes w) = \tau v \otimes \tau w, \qquad v \in T^m V^n,\ w \in W^n,\ \tau \in G_n .$$

It follows from (3.7), (3.8), and (3.9) that

$$\begin{aligned}(T^m V^n \otimes W^n)(x, \tau) &= \operatorname{tr}(\tau \mid W^n) \prod_h V^n(x_h, \tau) \\ &= \prod_{i,j} (\psi(g_i) \prod_h V(x_h{}^j, g_i))^{a_{ij}}.\end{aligned} \tag{3.10}$$

If M is any G_n module we say that an element $a \in M$ is invariant if $\tau a = a$ for all $\tau \in G_n$. Let $I(M)$ denote the space of invariant elements of M. We suppress the dependence on the group G_n in our notation and also let $I(M)$ denote the space of invariant elements in a G module M. Sums over τ (over g) are understood to range over all $\tau \in G_n$ ($g \in G$). The group will be clear from the context. If dim M is finite, it follows from the orthogonality relations for the characters that

$$\dim I(M) = (1/|\, G_n \,|) \sum_\tau \operatorname{tr}(\tau \mid M). \tag{3.11}$$

If M is a graded (or multigraded) G_n module then $I(M)$ is a graded (multigraded) vector space with Poincaré series

$$I(M)(x) = (1/|\, G_n \,|) \sum_\tau M(x, \tau). \tag{3.12}$$

If $\sigma \in S_n$ let $\epsilon(\sigma)$ be the sign of σ. Since the set of all elements $(f, 1)$ is a normal subgroup of G_n with factor group isomorphic to S_n, we may view ϵ as a representation of G_n. We say that an element $a \in M$ is alternating if

$\tau a = \epsilon(\tau)\, a$ for all $\tau \in G_n$. Let $\hat{I}(M)$ denote the space of alternating elements of M. If dim M is finite then

$$\dim \hat{I}(M) = (1/|\,G_n\,|) \sum_{\tau} \epsilon(\tau) \operatorname{tr}(\tau \mid M). \tag{3.13}$$

If M is graded (or multigraded) then

$$\hat{I}(M)(x) = (1/|\,G_n\,|) \sum_{\tau} \epsilon(\tau)\, M(x, \tau). \tag{3.14}$$

For notational convenience we agree that $V^0 = \mathbf{C}$ so $I(T^m V^0)(x) = 1 = \hat{I}(T^m V^0)(x)$.

3.15. THEOREM. *Let G be a finite group. Let V be a graded G module and let W be a finite-dimensional G module. If κ is an m-tuple of nonnegative integers let*

$$e_\kappa = \dim I((T^m V \otimes W)_\kappa), \tag{3.16}$$

where I denotes invariants under G. Define formal power series Q_n and $\hat{Q}_n$ in indeterminates $x_1, \ldots, x_m$ by

$$\prod_\kappa (1 - x^\kappa t)^{-e_\kappa} = \sum_{n=0}^{\infty} Q_n t^n, \tag{3.17}$$

$$\prod_\kappa (1 + x^\kappa t)^{e_\kappa} = \sum_{n=0}^{\infty} \hat{Q}_n t^n, \tag{3.18}$$

where t is an indeterminate. Then Q_n is the Poincaré series of $I(T^m V^n \otimes W^n)$ and $\hat{Q}_n$ is the Poincaré series of $\hat{I}(T^m V^n \otimes W^n)$, where I and $\hat{I}$ denote invariant and alternating elements under $G_n = G \wr S_n$.

Proof. Let $\tau \in G_n$ and suppose τ has type $a = (a_{ij})$. Recall that we have chosen elements $g_1, \ldots, g_r$ to represent the conjugacy classes of G. Let c_i be the order of the centralizer of g_i in G. Then [10, Sect. 3.9] the order of the centralizer of τ in G_n is

$$c_a = \prod_{i,j} a_{ij}!\,(jc_i)^{a_{ij}}. \tag{3.19}$$

Let ψ be the character of G afforded by W. It follows from (3.10), (3.12), and the remarks following (2.5) that

$$\begin{aligned} I(T^m V^n \otimes W^n)(x) &= (1/|\,G_n\,|) \sum_{\tau} (T^m V^n \otimes W^n)(x, \tau) \\ &= \sum_{a \in \mathscr{A}_n} (1/c_a) \prod_{i,j} \Big(\psi(g_i) \prod_h V(x_h^j, g_i)\Big)^{a_{ij}}. \end{aligned}$$

Let

$$s_{ij} = \frac{\psi(g_i)}{c_i} \prod_h V(x_h{}^j, g_i), \qquad p_a = \prod_{i,j} \frac{1}{a_{ij}!} \left(\frac{s_{ij}}{j}\right)^{a_{ij}}.$$

Then

$$I(T^m V^n \otimes W^n)(x) = \sum_{a \in \mathscr{A}_n} p_a . \tag{3.20}$$

Let $\mathscr{B}_n$ be the set of all sequences $b = (b_1, b_2, \ldots)$ of nonnegative integers such that $\sum jb_j = n$. Let

$$p_{i,b} = \prod_j (1/b_j!)(s_{ij}/j)^{b_j}, \qquad q_{i,n} = \sum_{b \in \mathscr{B}_n} p_{i,b} .$$

Let $\mathscr{A}_{n_1,\ldots,n_r}$ be the set of all $a \in \mathscr{A}_n$ such that the ith row of a is in $\mathscr{B}_{n_i}$ for $i = 1,\ldots, r$. We may identify $\mathscr{A}_{n_1,\ldots,n_r}$ with $\mathscr{B}_{n_1} \times \cdots \times \mathscr{B}_{n_r}$. Then

$$\mathscr{A}_n = \bigcup_{n_1+\cdots+n_r=n} \mathscr{A}_{n_1,\ldots,n_r} = \bigcup_{n_1+\cdots+n_r=n} \mathscr{B}_{n_1} \times \cdots \times \mathscr{B}_{n_r},$$

where the union is taken over all r-tuples $(n_1, \ldots, n_r)$ of nonnegative integers such that $n_1 + \cdots + n_r = n$. If $a \in \mathscr{B}_{n_1} \times \cdots \times \mathscr{B}_{n_r}$ then $p_a = p_{1,a_1} \cdots p_{r,a_r}$, where a_i is the ith row of a. Thus

$$\sum_{a \in \mathscr{A}_n} p_a = \sum_{n_1+\cdots+n_r=n} \sum_{a \in \mathscr{B}_{n_1} \times\cdots\times \mathscr{B}_{n_r}} p_{1,a_1} \cdots p_{r,a_r}$$

$$\sum_{n_1+\cdots+n_r=n} q_{1,n_1} \cdots q_{r,n_r} .$$

Setting

$$F_i(t) = \sum_{n=0}^{\infty} q_{i,n} t^n, \qquad i = 1,\ldots, r$$

we have

$$\sum_{n=0}^{\infty} I(T^m V^n \otimes W^n)(x)\, t^n = F_1(t) \cdots F_r(t).$$

Since

$$F_i(t) = \sum_{n=0}^{\infty} \left(\sum_{b \in \mathscr{B}_n} \prod_{j=1}^{n} (1/b_j!)(s_{ij}/j)^{b_j} \right) t^n$$

$$= \exp\left(\sum_{j=1}^{\infty} (s_{ij}/j)\, t^j \right)$$

we have

$$\sum_{n=0}^{\infty} I(T^m V^n \otimes W^n)(x)\, t^n = \exp\left(\sum_{i=1}^{r} \sum_{j=1}^{\infty} (s_{ij}/j)\, t^j \right).$$

It follows from (3.10) with $n = 1$ or from (3.5) that

$$(T^m V \otimes W)(x, g) = \psi(g) \prod_h V(x_h, g), \qquad g \in G,$$

and thus

$$\begin{aligned} \sum_{i=1}^{r} s_{ij} &= \sum_i (\psi(g_i)/c_i) \prod_h V(x_h{}^j, g_i) \\ &= (1/|G|) \sum_g \psi(g) \prod_h V(x_h{}^j, g) \\ &= I(T^m V \otimes W)(x_1{}^j, \dots, x_m{}^j) \\ &= \sum_\kappa e_\kappa x^{j\kappa}, \end{aligned} \tag{3.21}$$

where we write $j\kappa = (jk_1, \dots, jk_m)$ if $\kappa = (k_1, \dots, k_m)$. Thus

$$\begin{aligned} \sum_{i=1}^{r} \sum_{j=1}^{\infty} (s_{ij}/j)\, t^j &= \sum_\kappa e_\kappa \sum_{j=1}^{\infty} ((x^\kappa t)^j/j) \\ &= \log \prod_\kappa (1 - x^\kappa t)^{-e_\kappa} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} I(T^m V^n \otimes W^n)(x)\, t^n &= \exp \log \prod_\kappa (1 - x^\kappa t)^{-e_\kappa} \\ &= \prod_\kappa (1 - x^\kappa t)^{-e_\kappa}. \end{aligned}$$

This proves (3.17). The proof of (3.18) is similar. Let $\delta_j = (-1)^{j+1}$. If $\tau = (f, \sigma)$ has type (a_{ij}) then σ has $\sum_i a_{ij}$ cycles of length j so $\epsilon(\tau) = \prod_{i,j} \delta_j^{a_{ij}}$. Let

$$\hat{s}_{ij} = \frac{\delta_j \psi(g_i)}{c_i} \prod_h V(x_h{}^j, g_i), \qquad \hat{p}_a = \prod_{i,j} \frac{1}{a_{ij}!} \left(\frac{\hat{s}_{ij}}{j}\right)^{a_{ij}}.$$

Then (3.20) is replaced by

$$\hat{I}(T^m V^n \otimes W^n) = \sum_{a \in \mathscr{A}_n} \hat{p}_a,$$

(3.21) is replaced by

$$\sum_{i=1}^{r} \hat{s}_{ij} = \delta_j \sum_\kappa e_\kappa x^{j\kappa},$$

and (3.22) is replaced by

$$\sum_{i=1}^{r} \sum_{j=1}^{\infty} (\hat{s}_{ij}/j)\, t^j = -\sum_\kappa \sum_{j=1}^{\infty} ((-x^\kappa t)^j/j). \quad \square$$

4. Gordon's Theorem and Its Relatives

Let $A = \mathbf{C}[X_1, ..., X_p]$ be a polynomial algebra. Let G be a finite group of $\mathbf{C}$ algebra automorphisms of A. In this section we apply the formulas of Section 3 to the graded G module $A = \bigoplus_k A_k$ where A_k is the space of homogeneous polynomials of degree k. The main result, Theorem 4.12, contains Gordon's theorem [9, Sect. 2] on the coefficients in the expansion of $\prod_\kappa (1 - x^\kappa t)^{-1}$, as the special case $p = 1$ and $G = 1$.

We say that elements $Y_1, ..., Y_p$ of A are a basic set for (G, A) if each Y_j is homogeneous, G invariant, and A is integral over $\mathbf{C}[Y_1, ..., Y_p]$. The degrees $d_1, ..., d_p$ of the elements in a basic set are not uniquely determined by (G, A) since, for example, if $Y_1, ..., Y_p$ is a basic set then so is $Y_1^{k_1}, ..., Y_p^{k_p}$ for any positive integers $k_1, ..., k_p$. Theorem 4.12 and its corollaries are strongest when the degrees are small. Corollary 4.4 shows that we always have $d_1 \cdots d_p \equiv 0 \bmod |G|$. This gives a lower bound, which is in fact attained if and only if G, in its action on $\mathbf{C}X_1 + \cdots + \mathbf{C}X_p$, is generated by unitary reflections [3, p. 115, Théorème 4].

4.1. Lemma. *There exists a basic set for* (G, A).

Proof Since G is finite we know [2, Sect. 1.9, p. 323] that A is integral over $I(A)$, that $I(A)$ has transcendence degree p over $\mathbf{C}$, and that $I(A)$ is a finitely generated $\mathbf{C}$ algebra. The generators of $I(A)$ may be chosen homogeneous. The lemma is thus a consequence of the Noether normalization theorem [25, p. 200]. □

4.2. Lemma. *Suppose $Y_1, ..., Y_p$ is a basic set for (G, A) Let $B = \mathbf{C}[Y_1, ..., Y_p]$. Let $d_j = \deg(Y_j)$. Then the Poincaré series of B is*

$$B(x) = \prod_{j=1}^{p} (1 - x^{d_j})^{-1}.$$

Proof. Clear since $Y_1, ..., Y_p$ are algebraically independent over $\mathbf{C}$. □

4.3. Lemma. *Suppose $Y_1, ..., Y_p$ is a basic set for (G, A). Let $B = \mathbf{C}[Y_1, ..., Y_p]$. Then there exists a finite-dimensional G submodule C of A which is spanned by homogeneous elements and such that $A \simeq B \otimes C$ as graded G modules.*

Proof. We argue as in [3, p. 107, Théorème 2]. Since A is integral over B the ideal $L = AY_1 + \cdots + AY_p$ has finite codimension in A. Let $L_k = L \cap A_k$. Since L is a submodule of the semisimple G module A_k there exists a submodule C_k of A_k such that $A_k = L_k \oplus C_k$. Let $C = \bigoplus_k C_k$. Then C is a G module spanned by homogeneous elements and $A = L \oplus C$. Since A is integral over B it follows [3, p. 137, Exercice 5] that A is a free B module. Let $Z_1, Z_2, \ldots$ be a $\mathbf{C}$ basis for C consisting of homogeneous elements. Then by [3, p. 106, Remarque 1] $Z_1, Z_2, \ldots,$ is a basis for A as B module and thus $A \simeq B \otimes C$. □

4.4. COROLLARY. *Let $d_1, \ldots, d_p$ be the degrees of a basic set for (G, A) and let C be as in Lemma 4.3. Then $d_1 \cdots d_p \equiv 0 \bmod |G|$ and C affords a direct sum of copies of the regular representation of G.*

Proof. Since $A \simeq B \otimes C$ and B consists of G invariants we have

$$A(x, g) = B(x)\, C(x, g), \qquad g \in G. \tag{4.5}$$

Molien's formula [3, p. 110, Lemme 2] states that

$$A(x, g) = \det(1 - xg)^{-1}, \tag{4.6}$$

where det refers to the action of G on $\mathbf{C}X_1 + \cdots + \mathbf{C}X_p$. The module C affords a character, say γ, of G. It follows from (4.2) that $B(x)$ has the form $1/(1-x)^p F(x)$ where $F(x)$ is a polynomial and $F(1) = d_1 \cdots d_p$. Setting $x = 1$ we see from (4.5) and (4.6) that

$$\gamma(g) = C(1, g) = \begin{cases} d_1 \cdots d_p & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Write $\gamma = \sum_\psi c_\psi \psi$ where the c_ψ are integers and the sum is over all irreducible characters ψ of G. Then

$$|G|\, c_\psi = \sum_g \psi(g)\, \gamma(g^{-1}) = d_1 \cdots d_p \psi(1).$$

This formula shows, on taking $\psi = 1$ the principal character, that $d_1 \cdots d_p = c_1 |G|$. It also shows that $|G|\, \gamma = d_1 \cdots d_p \sum_\psi \psi(1)\, \psi$ is a multiple of the character of the regular representation. □

4.8. *Remarks.* If G, in its action on $\mathbf{C}X_1 + \cdots + \mathbf{C}X_p$, is generated by unitary reflections, then theorems of Chevalley [3, p. 107, Théorème 2] and Shephard–Todd–Chevalley [3, p. 115, Théorème 4] assert that one may choose a basic set $Y_1, \ldots, Y_p$ such that $I(A) = \mathbf{C}[Y_1, \ldots, Y_p] = B$ and then C affords the regular representation of G.

Corollary 4.4 is a refinement of Burnside's theorem on tensor powers of a faithful representation of a finite group [5, p. 299, Theorem 4]. To see this, let E be a vector space of finite dimension and let G be a finite subgroup of $GL(E)$. Both the tensor algebra $T(E)$ and the symmetric algebra $S(E)$ are $GL(E)$ modules and hence are G modules. If $X_1, \ldots, X_p$ is a basis for E then we may identify $S(E)$ with $A = \mathbf{C}[X_1, \ldots, X_p]$. Corollary 4.4 assures us that every simple G module is isomorphic to a submodule of $S(E)$ and hence, since we are in characteristic zero, to a submodule of $T(E)$.

Fix a positive integer n and let $A^n = T^n A$ be the nth tensor power of A. The $\mathbf{C}$ algebra structure in A defines a $\mathbf{C}$ algebra structure in A^n. For each $i = 1, \ldots, n$ define a $\mathbf{C}$ algebra monomorphism $\varphi_i : A \to A^n$ by

$$\varphi_i(s) = 1 \otimes \cdots \otimes s \otimes \cdots \otimes 1, \qquad s \in A, \tag{4.9}$$

where s appears in the ith position. Let $Z_{ij} = \varphi_i(X_j)$. Then $A^n = \mathbf{C}[\ldots, Z_{ij}, \ldots]$ is a polynomial algebra in np indeterminates. If $s \in A$ and $k = 1, \ldots, n$ let

$$a_k(s) = \sum_{i_1 < \cdots < i_k} \varphi_{i_1}(s)\, \varphi_{i_2}(s) \cdots \varphi_{i_k}(s),$$

where the sum is over all increasing sequences of k integers between 1 and n. Give A^n the grading (2.3). If G is a group of $\mathbf{C}$ algebra automorphisms of A, give A^n the G_n module structure (2.4). Then G_n acts as a group of $\mathbf{C}$ algebra automorphisms of A^n.

4.10. Lemma. *Let $Y_1, \ldots, Y_p$ be a basic set for (G, A). Then the elements $a_i(Y_j)$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant p$ are a basic set for (G_n, A^n).*

Proof. Since the Y_j are homogeneous, so are the $a_i(Y_j)$. If $s \in A$ then $a_k(s)$ is invariant under all elements $(1, \sigma)$ where $\sigma \in S_n$. If $s \in I(A)$ is invariant under G then $\varphi_i(s)$ is invariant under all $(f, 1)$ where f is any mapping of Ω_n into G. Thus $a_i(Y_j)$ is invariant under all elements $(1, \sigma)$ and $(f, 1)$ and hence is invariant under G_n.

Let $B^n = \mathbf{C}[\ldots, a_i(Y_j), \ldots]$. Since A is integral over B it follows that $\varphi_i(A)$ is integral over $\varphi_i(B)$ and thus A^n is integral over $\mathbf{C}[\ldots, \varphi_i(Y_j), \ldots]$. Let T be an indeterminate. Then

$$\prod_{i=1}^{n} (T - \varphi_i(Y_j)) = \sum_{i=0}^{n} (-1)^i\, a_i(Y_j)\, T^{n-i}$$

so $\varphi_i(Y_j)$ is integral over B^n. Thus A^n is integral over B^n. $\square$

4.11. Corollary. *The Poincaré series of B^n is*

$$B^n(x) = \prod_{i=1}^{n} \prod_{j=1}^{p} (1 - x^{i d_j})^{-1}.$$

Proof. Since A^n is integral over B^n it follows that B^n has transcendence degree np over $\mathbf{C}$. Thus the $a_i(Y_j)$ are algebraically independent. $\square$

4.12. THEOREM. *Let $A = \mathbf{C}[X_1, ..., X_p]$ be a polynomial algebra and let G be a finite group of $\mathbf{C}$ algebra automorphisms of A. Let $Y_1, ..., Y_p$ be a basic set for (G, A) and let $d_j = \deg(Y_j)$. Let W be a G module of finite dimension. For any m-tuple κ of nonnegative integers let $e_\kappa = \dim I((T^m A \otimes W)_\kappa)$. Define formal power series Q_n and P_n in indeterminates $x_1, ..., x_m$ by*

$$\prod_{\kappa} (1 - x^\kappa t)^{-e_\kappa} = \sum_{n=0}^{\infty} Q_n t^n, \tag{4.13}$$

$$P_n = Q_n \prod_{h=1}^{m} \prod_{i=1}^{n} \prod_{j=1}^{p} (1 - x_h^{id_j}). \tag{4.14}$$

Then P_n is a polynomial with nonnegative integer coefficients.

Proof. Let $B^n = \mathbf{C}[..., a_i(Y_j), ...]$. Since the elements $a_i(Y_j)$ are a basic set for (G_n, A^n) it follows from Lemma 4.3 with the pair (G, A) replaced by (G_n, A^n) that there exists a finite-dimensional G_n submodule C^n of A^n which is spanned by homogeneous elements and such that $A^n \simeq B^n \otimes C^n$ as graded G_n modules. Thus

$$T^m A^n \otimes W^n \simeq T^m B^n \otimes T^m C^n \otimes W^n,$$

where $W^n = T^n W$. Since B^n consists of G_n invariants, so does $T^m B^n$. Thus there is an isomorphism

$$I(T^m A^n \otimes W^n) \simeq T^m B^n \otimes I(T^m C^n \otimes W^n)$$

of graded vector spaces and a corresponding formula

$$I(T^m A^n \otimes W^n)(x) = (T^m B^n)(x) \cdot I(T^m C^n \otimes W^n)(x) \tag{4.15}$$

for the Poincaré series in a single indeterminate x. It follows from (4.11) that the Poincaré series of $T^m B^n$ is

$$(T^m B^n)(x) = \prod_{h=1}^{m} \prod_{i=1}^{n} \prod_{j=1}^{p} (1 - x_h^{id_j})^{-1}, \tag{4.16}$$

where x now stands for $(x_1, ..., x_m)$. It follows from the definition of P_n and Theorem 3.15 that

$$P_n = I(T^m C^n \otimes W^n)(x) \tag{4.17}$$

is the Poincaré series of a finite-dimensional multigraded vector space, and its coefficients are thus nonnegative integers. $\square$

If we define formal power series $\hat{Q}_n$ and $\hat{P}_n$ by

$$\prod_{\kappa} (1 + x^{\kappa} t)^{e_\kappa} = \sum_{n=0}^{\infty} \hat{Q}_n t^n, \tag{4.18}$$

$$\hat{P}_n = \hat{Q}_n \prod_{h=1}^{m} \prod_{i=1}^{n} \prod_{j=1}^{p} (1 - x_h^{id_j}), \tag{4.19}$$

then the same argument shows that

$$\hat{P}_n = \hat{I}(T^m C^n \otimes W^n)(x) \tag{4.20}$$

is also a polynomial with nonnegative integer coefficients. If $G = 1$ then Carlitz [6] proved for $m = 2$ and Wright [24] proved for any m that

$$\hat{Q}_n(x_1, \ldots, x_l^{-1}, \ldots, x_m) = (-1)^n x_l{}^n Q_n(x_1, \ldots, x_l, \ldots, x_m) \tag{4.21}$$

and thus

$$\hat{P}_n(x_1, \ldots, x_l, \ldots, x_m) = x_l^{n(n-1)/2} P_n(x_1, \ldots, x_l^{-1}, \ldots, x_m) \tag{4.22}$$

for all $l = 1, \ldots, m$. Thus the P_n determine the $\hat{P}_n$. We show following (4.30) that there are corresponding formulas for certain groups G. In (4.23)–(4.30) we confine our attention to the P_n and simply remark here that there are corresponding formulas for the $\hat{P}_n$.

4.23. Corollary. *The polynomials P_n have degree at most $\sum_{i=1}^{n} \sum_{j=1}^{p} (id_j - 1)$ in each of the indeterminates $x_1, \ldots, x_m$.*

Proof. Since $A^n \simeq B^n \otimes C^n$ it follows from (4.11) that

$$C^n(x) = \prod_{i=1}^{n} \prod_{j=1}^{p} (1 + x + \cdots + x^{id_j - 1}). \quad \square \tag{4.24}$$

In Section 5 we shall compute the P_n explicitly in certain cases, and this next corollary gives us a check on the computations.

4.25. Corollary. *For $1 \leqslant l \leqslant m$ define a polynomial $P_n{}^l$ in $m - 1$ indeterminates by*

$$P_n{}^l = P_n(x_1, \ldots, x_{l-1}, 1, x_{l+1}, \ldots, x_m).$$

Then

$$|G|^n P_n{}^l = (\dim W)^n (d_1 \cdots d_p)^n (n!)^{p-1} \prod_{h \neq l} C^n(x_h).$$

Proof. It follows from (4.17) and from (3.10) with C^n in place of V^n that

$$| G_n | P_n^l = \sum_{\tau} \operatorname{tr}(\tau \mid W^n) C^n(1, \tau) \prod_{h \neq l} C^n(x_h, \tau).$$

It follows from (4.7) with G_n in place of G that $C^n(1, \tau) = 0$ for $\tau \neq 1$ and thus $| G_n | P_n^l = (\dim W)^n C^n(1) \prod_{h \neq l} C^n(x_h)$. The assertion follows since $C^n(1) = (d_1 \cdots d_p)^n (n!)^p$ by (4.24), and $| G_n | = | G |^n n!$. $\square$

Theorem 4.12 may be used to prove a generalization of Gordon's theorem which does not involve any group invariants in its statement.

4.26. THEOREM. *Let r be a positive integer and let s be an integer. If $\kappa = (k_1, \ldots, k_m)$ is an m-tuple of nonnegative integers let $\| \kappa \| = k_1 + \cdots + k_m$. Define formal power series Q_n and P_n in indeterminates $x_1, \ldots, x_m$ by*

$$\prod_{\|\kappa\| \equiv s \bmod r} (1 - x^\kappa t)^{-1} = \sum_{n=0}^{\infty} Q_n t^n,$$

$$P_n = Q_n \prod_{h=1}^{m} \prod_{i=1}^{n} (1 - x_h^{ir}).$$

Then P_n is a polynomial with nonnegative integer coefficients.

Proof. If $r = 1$, in which case s has no significance, this is Gordon's theorem. Let $G = \langle g \rangle$ be a cyclic group of order r. Let $\zeta \in \mathbf{C}$ be a primitive rth root of unity. Let $A = \mathbf{C}[X]$ be a polynomial algebra in one indeterminate. Let G act on A by $gX^k = \zeta^k X^k$. Then $I(A) = \mathbf{C}[X^r]$ and we may choose $\{X^r\}$ as a basic set for (G, A). Let W be the one-dimensional vector space $\mathbf{C}$ with the G module structure $g \cdot 1 = \zeta^{-s}$. For $h = 1, \ldots, m$ define $U_h \in T^m A$ by $U_h = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1$ where X appears in the hth position. Then $T^m A = \mathbf{C}[U_1, \ldots, U_m]$ is a polynomial algebra in m indeterminates, multigraded according to (3.2) by $(T^m A)_\kappa = \mathbf{C} U^\kappa$ where $U^\kappa = U_1^{k_1} \cdots U_m^{k_m}$. The G module structure (3.3) is given by $gU^\kappa = \zeta^{\|\kappa\|} U^\kappa$. The G module structure on $(T^m A \otimes W)_\kappa$ is thus given by

$$g(U^\kappa \otimes 1) = \zeta^{\|\kappa\| - s}(U^\kappa \otimes 1)$$

so that

$$(T^m A \otimes W)(x, g) = \sum_{\kappa} \zeta^{\|\kappa\| - s} x^\kappa.$$

It follows that

$$I(T^m A \otimes W)(x) = (1/r) \sum_{i=1}^{r} \zeta^{i(\|\kappa\| - s)} x^\kappa = \sum_{\|\kappa\| \equiv s \bmod r} x^\kappa.$$

Thus $e_\kappa = 1$ if $\| \kappa \| \equiv s \bmod r$ and $e_\kappa = 0$ otherwise. The assertion follows now from Theorem 4.12. $\square$

4.27. *Remarks.* Let G and A be as in the proof of Theorem 4.26. For $i = 1,..., n$ define $X_i \in A^n$ as in (4.9) by $X_i = \varphi_i(X) = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1$ where X appears in the ith position. Do not confuse X_i with the element $U_i \in A^m$ defined in the proof of (4.26). The group G_n acts on the first homogeneous component $A_1{}^n = \mathbf{C}X_1 + \cdots + \mathbf{C}X_n$ of A^n according to (2.4). Thus

$$(1, \sigma)\, X_i = X_{\sigma i}\,, \qquad (f, 1)\, X_i = f(i)\, X_i\,.$$

The corresponding group of n by n matrices consists of all monomial matrices whose nonzero entries are rth roots of unity. It is a unitary reflection group. The algebra $B^n = \mathbf{C}[a_1(X^r),..., a_n(X^r)]$, defined in the proof of Lemma 4.10, consists of all symmetric polynomials in $X_1{}^r,..., X_n{}^r$ and thus [16, Sect. 6(2)] $B^n = I(A^n)$. Let L^n be the ideal of A^n generated by the homogeneous elements of positive degree in B^n. Work of Steinberg [20, Theorems 1 and 2] and, in the special case $r = 1$, Fischer–Schur [8, Sect. 5], shows that the space C^n spanned by the polynomial[2]

$$J_n = (X_1 \cdots X_n)^{r-1} \prod_{1 \leqslant i < j \leqslant n} (X_i{}^r - X_j{}^r) \tag{4.28}$$

and its partial derivatives of all orders, is a complement to L^n in A^n and thus $A^n \simeq B^n \otimes C^n$ by (4.3). In case $r = 1$ this space C^n is the space H^n of S_n-harmonic polynomials mentioned in the introduction to this paper. In case $r = 1$, Corollary 4.23 says that P_n has degree at most $n(n-1)/2$ in each of the indeterminates x_i. Wright [24, Sect. 3] proved that the bound is attained. The element J_n yields a different sort of bound. Since J_n is alternating it follows that $J_n \otimes \cdots \otimes J_n$, m factors, is symmetric if m is even and alternating if m is odd. Since J_n is, up to a constant multiple, the unique homogeneous element of maximal degree in C^n it follows from (4.17) that P_n has total degree $mn(n-1)/2$ precisely when m is even and $\hat{P}_n$ has total degree $mn(n-1)/2$ precisely when m is odd.

Finally, note in the situation of Theorem 4.26 that (4.25) becomes

$$P_n{}^l = \prod_{h \neq l} \prod_{i=1}^{n} (1 + x_h + \cdots + x_h^{ir-1}). \tag{4.29}$$

In particular

$$P_n(1,..., 1) = (r^n n!)^{m-1}, \tag{4.30}$$

a formula of Carlitz and Wright in case $r = 1$.

We return now to the case of an arbitrary finite group G acting as a group of automorphisms of a polynomial algebra $A = \mathbf{C}[X_1,..., X_p]$. We will show, with some hypothesis on the action of G, that there are formulas analogous

[2] There is a misprint in the formula for J_n given on the last line of [20, p. 394].

to (4.21) and (4.22) which have their origin in Molien's formula (4.6). It follows from (4.6) that

$$A(x^{-1}, g^{-1}) = (-1)^p x^p \det(g) A(x, g), \qquad g \in G,$$

where det denotes the determinant of g as linear transformation of $A_1 = \mathbf{C}X_1 + \cdots + \mathbf{C}X_p$. Let $\tau = (f, \sigma) \in G_n$. The map $\theta\colon G_n \to \mathbf{C}^\times$ defined by

$$\theta(\tau) = \prod_{p \in \Omega_n} \det f(p)$$

is a homomorphism of groups. If τ has type (a_{ij}) then

$$\theta(\tau) = \prod_{i,j} \det(g_i)^{a_{ij}}.$$

Let $b = \sum_{i,j} a_{ij}$. Since $n = \sum_{i,j} j a_{ij}$ we have $\epsilon(\tau) = (-1)^{n+b}$. It follows from (2.6) and the fact that $\bar{\lambda} = \lambda^{-1}$ for any eigenvalue λ of τ or g_i that

$$\begin{aligned} A^n(x^{-1}, \tau^{-1}) &= (-1)^{bp} x^{np} \theta(\tau) A^n(x, \tau) \\ &= (-1)^{np} x^{np} \epsilon(\tau)^p \theta(\tau) A^n(x, \tau). \end{aligned}$$

Assume now that p is odd and that $\det g = 1$ for all $g \in G$. Then $\theta(\tau) = 1$ for all τ and thus

$$A^n(x^{-1}, \tau^{-1}) \epsilon(\tau^{-1}) = (-1)^n x^{np} A^n(x, \tau).$$

It follows from (3.7) by averaging over the group G_n that

$$\hat{I}(T^m A^n)(x_1, \ldots, x_l^{-1}, \ldots, x_m) = (-1)^n x_l^{np} I(T^m A^n)(x_1, \ldots, x_l, \ldots, x_m)$$

for any $l = 1, \ldots, m$. If $W = \mathbf{C}$ with trivial G action, this is a generalization of (4.21) and leads to an analog of (4.22). I do not know if one can prove such formulas without the assumption that p is odd and $\det(g) = 1$.

5. Character Formulas

Let $X_1, \ldots, X_n$ be indeterminates and let $s_i = X_1^i + \cdots + X_n^i$ for $i = 1, 2, 3, \ldots$. Since $s_1, \ldots, s_n$ are algebraically independent there exist unique polynomials $p_l = p_l(s_1, \ldots, s_n)$ such that

$$\frac{1}{(1 - X_1 t) \cdots (1 - X_n t)} = \sum_{l \geq 0} p_l t^l, \tag{5.1}$$

where t is an indeterminate. We agree to write $p_l = 0$ if $l < 0$. If λ is a partition of some integer, with parts $\lambda_1 \geq \lambda_2 \geq \cdots$ let

$$\Phi_\lambda(s_1, \ldots, s_n) = \det(p_{\lambda_i - i + j}) \tag{5.2}$$

be the corresponding Schur function [18] where the size of the determinant is equal to the number of parts of λ.

In this section we derive a formula for the series Q_n as a sum indexed by irreducible characters of G_n, in which the summands are Schur functions. If G is cyclic and $A = \mathbf{C}[X]$ these Schur functions may be computed explicitly and we arrive at a formula for P_n in terms of the hooks of certain partition diagrams. To avoid unnecessary complication we assume that the module W of (3.15) is $W = \mathbf{C}$ with trivial G action so that $Q_n = I(T^m V^n)(x)$.

We begin by summarizing the character theory of a wreath product $G_n = G \wr S_n$ as given by Specht [17] in his thesis. Kerber [10] has simplified part of Specht's argument by using Clifford's theorem on characters induced from a normal subgroup. There is a recent exposition by Williams [22] of Specht's formula for the characters of G_n in terms of Schur functions.

Let $\psi_1, \ldots, \psi_r$ be the irreducible characters of G. The irreducible characters of G_n are indexed by r-tuples $(\lambda_1, \ldots, \lambda_r)$, where λ_k is a partition of a non-negative integer n_k and the integers n_k satisfy $n_1 + \cdots + n_r = n$. The character χ corresponding to $(\lambda_1, \ldots, \lambda_r)$ may be constructed as follows. Let V_k be a G module which affords ψ_k. Then $T^{n_k} V_k$ has the structure of G_{n_k} module given by (2.4) and affords a character, say ξ_k, of G_{n_k}. Let φ_{λ_k} be the character of S_{n_k} corresponding to the partition λ_k. Since G_{n_k} has S_{n_k} as a factor group, characters of S_{n_k} may be viewed as characters of G_{n_k}. Thus $\xi_k \varphi_{\lambda_k}$ is a character of G_{n_k}. We may view the direct product $G_{n_1} \times \cdots \times G_{n_r}$ as a subgroup of G_n. The character χ of G_n is induced by the character $\xi_1 \varphi_{\lambda_1} \# \cdots \# \xi_r \varphi_{\lambda_r}$ of the subgroup $G_{n_1} \times \cdots \times G_{n_r}$, where $\#$ as in [7, Sect. 43] denotes the outer tensor product.

If $a = (a_{ij}) \in \mathscr{A}_n$ let χ_a be the value of the character χ on an element of type a and let c_a, given by (3.19), be the order of the centralizer of an element of type a. Let s_{ij} be indeterminates, where $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$. Write $s^a = \prod_{i,j} s_{ij}^{a_{ij}}$. The polynomial

$$\Phi_\chi = \sum_{a \in \mathscr{A}_n} (1/c_a)\, \chi_a s^a \tag{5.3}$$

is called the characteristic of the character χ. For $1 \leqslant k \leqslant r$ and $1 \leqslant j \leqslant n$ define

$$S_{k,j} = \sum_{i=1}^{r} (1/c_i)\, \psi_k(g_i)\, s_{ij}, \tag{5.4}$$

where c_i, g_i are as in (3.19). If χ corresponds to the r-tuple $(\lambda_1, \ldots, \lambda_r)$ of partitions then there is a remarkable formula [17, pp. 18–27] and [22, Theorem 2.2]

$$\Phi_\chi = \prod_{k=1}^{r} \Phi_{\lambda_k}(S_{k,1}, \ldots, S_{k,n}) \tag{5.5}$$

for the characteristic in terms of the Schur functions.

Let $V = \oplus_k V_k$ be a graded G module. If ψ is an irreducible character of G let V_ψ be the isotypic component of V of type ψ, by definition the sum of all simple submodules of V which afford ψ. Then V_ψ is a graded vector space and has a Poincaré series $V_\psi(x)$. Let ν_k be the character of G afforded by V_k and let (ψ, ν_k) be the multiplicity of ψ in ν_k. Then

$$V_\psi(x) = \psi(1) \sum_k (\psi, \nu_k)\, x^k. \tag{5.6}$$

Let $\tilde{V}_\psi(x) = V_\psi(x)/\psi(1)$. Then

$$V_\psi(x, g) = \psi(g)\, \tilde{V}_\psi(x), \qquad g \in G. \tag{5.7}$$

It follows from the orthogonality relations for the characters that

$$\tilde{V}_\psi(x) = (1/|G|) \sum_g \overline{\psi(g)}\, V(x, g). \tag{5.8}$$

Since $V = \oplus V_\psi$, summed over all irreducible characters ψ, we have

$$V(x, g) = \sum_\psi V_\psi(x, g) = \sum_\psi \psi(g)\, \tilde{V}_\psi(x). \tag{5.9}$$

If $V = A$ is a polynomial algebra on which G acts as a group of automorphisms, write $A^n \simeq B^n \otimes C^n$ as in the proof of Theorem 4.3; this presupposes the choice of a basic set for (G, A) which is fixed once and for all. There are formulas analogous to (5.6)–(5.9) with G replaced by G_n, ψ replaced by an irreducible character χ of G_n, and V replaced by a graded G_n module, for example V^n, A^n, or C^n. Thus $\tilde{V}_\chi{}^n(x)$ and $\tilde{C}_\chi{}^n(x)$ are defined.

5.10. THEOREM. *Let χ be the irreducible character of G_n which corresponds to the r-tuple $(\lambda_1, \ldots, \lambda_r)$ of partitions. Then*

$$\tilde{V}_\chi{}^n(x) = \prod_{k=1}^{r} \Phi_{\lambda_k}(\tilde{V}_{\psi_k}(x), \tilde{V}_{\psi_k}(x^2), \ldots, \tilde{V}_{\psi_k}(x^n))$$

and

$$B^n(x)\, \tilde{C}_\chi{}^n(x) = \prod_{k=1}^{r} \Phi_{\lambda_k}(\tilde{A}_{\psi_k}(x), \tilde{A}_{\psi_k}(x^2), \ldots, \tilde{A}_{\psi_k}(x^n)).$$

Proof. It follows from the analog of (5.8) for G_n and from Lemma 2.6 that

$$\tilde{V}_\chi{}^n(x) = \sum_a (1/c_a)\, \bar{\chi}_a \prod_{i,j} V(x^j, g_i)^{a_{ij}}.$$

Define a $\mathbf{C}$ algebra homomorphism θ: $\mathbf{C}[\ldots, s_{ij}, \ldots] \to \mathbf{C}[[x]]$ by $\theta(s_{ij}) = V(x^j, g_i)$. Extend complex conjugation to a semiautomorphism of $\mathbf{C}[\ldots,$

s_{ij} ,...] by letting it fix the s_{ij} qnd use a bar to denote this semiautomorphism. It follows from (5.3) that $\tilde{V}_\chi{}^n(x) = \theta(\overline{\Phi_\chi})$. From (5.4) and (5.8) we have

$$\theta(\overline{S_{k,j}}) = \sum_{i=1}^{r} (1/c_i)\, \overline{\psi_k(g_i)}\, V(x^j, g_i) = \tilde{V}_{\psi_k}(x^j).$$

The formula for $\tilde{V}_\chi{}^n(x)$ follows now from (5.5). If $V = A$ is a polynomial algebra then, since B^n consists of G_n invariants, we have $A_\chi{}^n \simeq B^n \otimes C_\chi{}^n$. It follows that $A_\chi{}^n(x) = B^n(x)\, C_\chi{}^n(x)$ and thus $\tilde{A}_\chi{}^n(x) = B^n(x)\, \tilde{C}_\chi{}^n(x)$. □

5.11. THEOREM. *If χ_1 ,..., χ_m are irreducible characters of G_n let*

$$c(\chi_1 ,..., \chi_m) = (1/|\, G_n\, |) \sum_{\tau} \chi_1(\tau) \cdots \chi_m(\tau).$$

Then

$$Q_n = \sum_{\chi_1} \cdots \sum_{\chi_m} c(\chi_1 ,..., \chi_m)\, \tilde{V}^n_{\chi_1}(x_1) \cdots \tilde{V}^n_{\chi_m}(x_m), \tag{5.12}$$

where each of χ_1 ,..., χ_m ranges over the set of irreducible characters of G_n . If $V = A$ is a polynomial algebra then

$$P_n = \sum_{\chi_1} \cdots \sum_{\chi_m} c(\chi_1 ,..., \chi_m)\, \tilde{C}^n_{\chi_1}(x_1) \cdots \tilde{C}^n_{\chi_m}(x_m). \tag{5.13}$$

Proof. The analog of (5.9) for the pair (G_n , V^n) is

$$V^n(x_h , \tau) = \sum_{\chi} \chi(\tau)\, \tilde{V}_\chi{}^n(x_h), \qquad h = 1,..., m,$$

where the sum ranges over all irreducible characters χ of G_n . Since we are assuming in this section that $W = \mathbf{C}$ with trivial G action, Theorem 3.15 says that $Q_n = I(T^m V^n)(x)$. Thus (5.12) follows from (3.7) and (3.12). We know from (4.17) that $P_n = I(T^m C^n)(x)$ and thus (5.14) follows in the same way. □

5.14. *Remarks.* Note that (5.10) and (5.11) taken together allow one to compute the Q_n and P_n in terms of Schur functions. Since the $\tilde{C}_\chi{}^n(x)$ are polynomials with nonnegative integer coefficients and the $c(\chi_1 ,..., \chi_m)$ are nonnegative integers, (5.13) is a refinement of Theorem 4.12. If $m = 2$ then the orthogonality relations for the characters may be used to simplify (5.12) and (5.13) and we get

$$Q_n = \sum_{\chi} \tilde{V}_\chi{}^n(x_1)\, \tilde{V}_{\bar{\chi}}{}^n(x_2), \tag{5.12$'$}$$

$$P_n = \sum_{\chi} \tilde{C}_\chi{}^n(x_1)\, \tilde{C}_{\bar{\chi}}{}^n(x_2), \tag{5.13$'$}$$

where the sums are over all irreducible characters of G_n. Suppose in particular that $G = 1$ and $A = \mathbf{C}[X]$. Then G_n is the symmetric group. The collapse of (5.12) into (5.12′) in this case is related to an observation of MacMahon [11, Sect. 301–302] who, although he did not have any formulas like (5.12) or (5.13), remarked nevertheless that the case $m = 3$ is "very complex and leads to no interesting results" while the case $m = 2$ "is, however, worth a moment's consideration." From the point of view of characters one sees good reason for his pessimism since $c(\chi_1, \chi_2, \chi_3)$ is the multiplicity of χ_3 in $\chi_1\chi_2$ and there is no closed formula for the decomposition of a product of irreducible characters of S_n. When $n = 3$ the symmetric group S_n has only three irreducible characters and one can use (5.13) to derive Wright's formula for $P_3(x_1, ..., x_m)$[24, Sect. 4]; the polynomials $\tilde{C}_\chi{}^3(x)$ are given in (5.21). One could surely push the calculations further.

In the rest of this section we let $G = \langle g \rangle$ be a cyclic group of order r, let $A = \mathbf{C}[X]$, and let G act on A by $gX^k = \zeta^k X^k$, where ζ is a primitive rth root of unity. This is the situation we studied in the proof of Theorem 4.23. We shall derive a formula for the $\tilde{C}_\chi{}^n(x)$ and hence, by (5.13), a formula for P_n which allows us to compute in concrete cases. Define $X_i \in A^n$ as in (4.27) by $X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1$, where X appears in the ith position. Then $A^n = \mathbf{C}[X_1, ..., X_n]$ is a polynomial algebra in n indeterminates, graded according to (2.3) so that $A_k{}^n$ is the space of homogeneous polynomials of degree k. The irreducible characters $\psi_1, ..., \psi_r$ of G are given by $\psi_k(g) = \zeta^{k-1}$. Since $I(A) = \mathbf{C}[X^r]$ and $A_{\psi_k} = I(A)\, X^{k-1}$ we have

$$\tilde{A}_{\psi_k}(x) = A_{\psi_k}(x) = x^{k-1}/(1 - x^r). \tag{5.15}$$

If k is a positive integer write $(\mathbf{k}) = 1 - x^k$ and $(\mathbf{k})! = (\mathbf{1})(\mathbf{2}) \cdots (\mathbf{k})$. If λ is a partition of n with parts $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ let

$$F_\lambda(x) = x^d/(\mathbf{h_1}) \cdots (\mathbf{h_n}), \tag{5.16}$$

where $d = \sum i\lambda_i$ and the h_i are the hook lengths of λ. Formula (57) of [18] may be rewritten as

$$x^{-n}F_\lambda(x) = \det(1/(\boldsymbol{\lambda}_\mathbf{i} - \mathbf{i} + \mathbf{j})!), \tag{5.17}$$

where we agree that $(\mathbf{0})! = 1$ and $1/(\mathbf{k})! = 0$ if k is negative.

5.18. THEOREM. *Suppose G is cyclic of order r and $A = \mathbf{C}[X]$. Let χ be the irreducible character of G_n which corresponds to the r-tuple $(\lambda_1, ..., \lambda_r)$ of partitions, where λ_k is a partition of n_k and $n_1 + \cdots + n_r = n$. Let $\lambda_{1k} \geqslant \lambda_{2k} \geqslant \cdots$ be the parts of λ_k and let $h_{1k}, h_{2k}, ...$ be the hook lengths of λ_k. Let*

$$d = \sum_{i,k} i\lambda_{ik}, \qquad e = \sum_k kn_k.$$

Let $C_\chi{}^n$ be the isotypic component of C^n of type χ and let $\tilde{C}_\chi{}^n(x) = C_\chi{}^n(x)/\chi(1)$. Then

$$\tilde{C}_\chi{}^n(x) = x^{dr+e-nr-n} \prod_{i=1}^{n} (\mathbf{ir}) \Big/ \prod_{k=1}^{r} \prod_{i=1}^{n_k} (\mathbf{h}_{ik}\mathbf{r}). \tag{5.19}$$

Proof. Let $p_l = p_l(s_1, ..., s_n)$ be as in (5.1). Since $s_1, ..., s_n$ are algebraically independent there exists for each $k = 1, ..., r$ a homomorphism $\theta_k : \mathbf{C}[s_1, ..., s_n] \to \mathbf{C}[[x]]$ of $\mathbf{C}$ algebras such that

$$\theta_k(s_i) = \tilde{A}_{\psi_k}(x^i), \qquad i = 1, ..., n.$$

Fix a positive integer $j \leqslant n$. Then [12, pp. 106–108]

$$p_j = \sum_a (1/c_a)\, s^a,$$

where $c_a = a_1!\, 1^{a_1} \cdots a_j!\, j^{a_j}$, $s^a = s_1^{a_1} \cdots s_j^{a_j}$ and the sum is over all nonnegative integer solutions $a = (a_1, ..., a_j)$ to $a_1 + 2a_2 + \cdots + ja_j = j$. Let $t_i = 1/(\mathbf{ir})$ and let $t^a = t_1^{a_1} \cdots t_j^{a_j}$. Then by (5.15)

$$\theta_k(p_j) = x^{j(k-1)} \sum_a (1/c_a)\, t^a.$$

Let $R_j = \mathbf{C}[X_1, ..., X_j]$. Let the symmetric group S_j act as a group of algebra automorphisms of R_j by permuting $X_1, ..., X_j$. Since $I(R_j)$ is generated by algebraically independent elements of degrees $1, 2, ..., j$ we have

$$I(R_j)(x) = 1/(\mathbf{j})!,$$

where I refers to S_j invariants. On the other hand, by Molien's formula (4.6) applied with R_j in place of A we have

$$I(R_j)(x) = \sum_a 1/c_a (\mathbf{1})^{a_1} (\mathbf{2})^{a_2} \cdots (\mathbf{j})^{a_j}.$$

Replacing x by x^r gives

$$\sum_a (1/c_a)\, t^a = 1/(\mathbf{r})(\mathbf{2r}) \cdots (\mathbf{jr}).$$

Thus

$$\theta_k(p_j) = x^{j(k-1)}/(\mathbf{r})(\mathbf{2r}) \cdots (\mathbf{jr}). \tag{5.20}$$

Let $d_k = \sum_i i\lambda_{ik}$. Fix k and write $\nu = \lambda_k$ and $\nu_i = \lambda_{ik}$. Then using (5.2), (5.20), (5.17), and (5.16) with ν in place of λ we get

$$\begin{aligned}
&\Phi_\nu(\tilde{A}_{\psi_k}(x), \tilde{A}_{\psi_k}(x^2),..., \tilde{A}_{\psi_k}(x^n)) \\
&= \theta_k(\Phi_\nu(s_1,..., s_n)) \\
&= \det(\theta_k(p_{\nu_i-i+j})) \\
&= \det(x^{(k-1)(\nu_i-i+j)}/(\mathbf{r})(\mathbf{2r}) \cdots ((\boldsymbol{\nu}_\mathbf{i} - \mathbf{i} + \mathbf{j})\, \mathbf{r})) \\
&= x^{(k-1-r)n_k} F_\nu(x^r) \\
&= x^{(k-1-r)n_k+rd_k} \Big/ \prod_{i=1}^{n_k} (\mathbf{h}_{\mathbf{ik}}\mathbf{r}).
\end{aligned}$$

It follows from Theorem 5.10 that

$$B^n(x)\, \tilde{C}_\chi{}^n(x) = x^{dr+e-nr-n} \Big/ \prod_{k=1}^{r} \prod_{i=1}^{n_k} (\mathbf{h}_{\mathbf{ik}}\mathbf{r}).$$

Since (4.11) says $B^n(x) = 1/(\mathbf{r})(\mathbf{2r}) \cdots (\mathbf{nr})$ the proof is complete. □

5.21. Example. Let $m = 2$, $n = 3$, and $r = 1$. Then $G_3 = S_3$ and χ is a character of S_3 corresponding to the partition λ of 3. The polynomials $\tilde{C}_\chi{}^3(x)$ are given by

λ	d	e	$\tilde{C}_\chi{}^3(x)$
3	3	3	1
21	4	3	$x + x^2$
1^3	6	3	x^3

According to (5.13′) we have

$$P_3 = 1 + (x_1 + x_1{}^2)(x_2 + x_2{}^2) + x_1{}^3 x_2{}^3.$$

When $m = 2$ and $r = 1$ the polynomials P_4, P_5, P_6 have been computed by Roselle [14] who proves Gordon's theorem in this case by interpreting the coefficients of the P_n in terms of rises and readings of permutations. Roselle states a formula [14, p. 146, line 9] which amounts to the case $r = 1$ of (5.13′). This formula is closely related to some work of Stanley [19, Sect. 5] on plane partitions.

5.22. Example. Let $m = 2$, $n = 3$, and $r = 2$. Then G in its action on $A_1{}^3$ is the group of all 3 by 3 monomial matrices with nonzero entries equal to ± 1. It is the symmetry group of the cube and has order 48. The character χ corresponds to a pair of partitions λ_1, λ_2 of integers with sum equal to 3. The polynomials $\tilde{C}_\chi^3(x)$ are given by

λ_1	λ_2	d	e	$\tilde{C}_\chi^3(x)$
3	ϕ	3	3	1
21	ϕ	4	3	$x^2 + x^4$
1^3	ϕ	6	3	x^6
2	1	3	4	$x + x^3 + x^5$
1^2	1	4	4	$x^3 + x^5 + x^7$
1	2	3	5	$x^2 + x^4 + x^6$
1	1^2	4	5	$x^4 + x^6 + x^8$
ϕ	3	3	6	x^3
ϕ	21	4	6	$x^5 + x^7$
ϕ	1^3	6	6	x^9

Since all characters of G are real we have $\chi = \bar{\chi}$ in (5.13′) and thus $P_3 = \sum_\chi \tilde{C}_\chi^3(x_1)\, \tilde{C}_\chi^3(x_2)$. Formula (4.26) serves as a check on the computations. In this case it says, setting $x = x_2$, that $P_3(1, x) = \mathbf{(2)(4)(6)/(1)(1)(1)}$.

6. The Polya–Redfield Counting Formula

In Section 2 we chose V to be a graded G module. Assume now that V is multigraded by the set of all m-tuples $\kappa = (k_1, \ldots, k_m)$ of nonnegative integers. Replace the definitions (2.1)–(2.3) by

$$V(x, g) = \sum_\kappa \operatorname{tr}(g \mid V_\kappa)\, x^\kappa, \tag{6.1}$$

$$V(x) = \sum_\kappa \dim(V_\kappa)\, x^\kappa, \tag{6.2}$$

$$V_\kappa^n = \bigoplus_{\kappa_1 + \cdots + \kappa_n = \kappa} V_{\kappa_1} \otimes \cdots \otimes V_{\kappa_n}, \tag{6.3}$$

where $x = (x_1, \ldots, x_m)$ and $x^\kappa = x_1^{k_1} \cdots x_m^{k_m}$. There is a formula

$$V^n(x, \tau) = \prod_{i,j} V(x^j, g_i)^{a_{ij}} \qquad \tau \in G_n \tag{6.4}$$

analogous to (2.6) where x^j means $(x_1^j, \ldots, x_m^j)$. The proof is the same as the proof of (2.6). In particular, if $\tau = (1, \sigma)$ then

$$V^n(x, \tau) = \prod V(x^j)^{a_j}, \tag{6.5}$$

where $a_j = a_j(\sigma)$ is the number of cycles in σ of length j. Let

$$e_\kappa = \dim V_\kappa . \tag{6.6}$$

Define formal power series Q_n and $\hat{Q}_n$ in indeterminates $x_1 ,..., x_m$ by

$$\prod_\kappa (1 - x^\kappa t)^{-e_\kappa} = \sum_n Q_n t^n, \tag{6.7}$$

$$\prod_\kappa (1 + x^\kappa t)^{e_\kappa} = \sum_n \hat{Q}_n t^n. \tag{6.8}$$

Polya [13, formulas 1.12 and 1.14] has shown, in a different language which does not involve the notion of graded vector space, that

$$Q_n = (1/n!) \sum_a (1/c_a)\, V(x)^{a_1}\, V(x^2)^{a_2} \cdots V(x^n)^{a_n}, \tag{6.9}$$

$$\hat{Q}_n = (1/n!) \sum_a (\epsilon_a/c_a)\, V(x)^{a_1}\, V(x^2)^{a_2} \cdots V(x^n)^{a_n}, \tag{6.10}$$

where $\epsilon_a = (-1)^{a_2+a_4+\cdots}$, $c_a = a_1!\, 1^{a_1} \cdots a_n!\, n^{a_n}$ and the sum is over all solutions $a = (a_1 ,..., a_n)$ to $\sum ja_j = n$. Polya's argument uses the logarithm–exponential trick as in the last lines of the proof of (3.17) and there is a formal similarity between (6.9), (6.10), and (3.17), (3.18), but I have not been able to exploit (6.9) and (6.10). The reasons are first, that there is group theoretic information contained in the e_κ of (3.16) while there is no group theory in the e_κ of (6.6); and second, that even in case $G = 1$ the formulas (6.9) and (6.10) do not lead to a proof of Gordon's theorem because one cannot apply the tensor product lemma (4.3).

In the hope that it may provide illumination for some reader, we show that (6.5) amounts to the Polya–Redfield counting formula [4,13] and hence that (6.4) is a generalization of this formula. Let R be a set. Suppose we are given a mapping w called the weight, from R into the set of all monomials x^κ. Let $R_\kappa = \{r \in R \mid w(r) = x\}$. Assume that R_κ is finite for all κ. The series $\sum_\kappa \mid R_\kappa \mid x^\kappa$ is called the figure counting series or store enumerator. Let V_κ be a vector space with basis R_κ and let $V = \bigoplus_\kappa V_\kappa$. The Poincaré series

$$V(x) = \sum_\kappa \dim(V_\kappa)\, x^\kappa = \sum_\kappa \mid R_\kappa \mid x^\kappa$$

of V is thus the figure counting series. Fix a positive integer n. Let $D = \{1,..., n\}$ and let F be the set of all functions from D to R. If $f \in F$ define the weight $w(f)$ of f by

$$w(f) = \prod_{d \in D} w(f(d)).$$

Let F_κ be the set of functions of weight x^κ. The group S_n permutes F_κ by $(\sigma f)(d) = f(\sigma^{-1}d)$. The mapping $f \to f(1) \otimes \cdots \otimes f(n)$ carries F into a basis for V^n and commutes with the action of S_n. This correspondence between functions and tensors has been exploited in connection with Polya's theorem by Williamson [23]. If $\sigma \in S_n$ let $F_{\kappa,\sigma} = \{f \in F_\kappa \mid \sigma f = f\}$ and let $F_\sigma = \{f \in F \mid \sigma f = f\}$. Then $| F_{\kappa,\sigma} | = \operatorname{tr}(\sigma \mid V_\kappa{}^n)$. Since F_σ is a disjoint union of the $F_{\kappa,\sigma}$ we have

$$\sum_{f \in F_\sigma} w(f) = V^n(x, \sigma).$$

It follows from (6.5) that

$$\sum_{f \in F_\sigma} w(f) = V(x)^{a_1} V(x^2)^{a_2} \cdots V(x^n)^{a_n}, \tag{6.11}$$

where $a_j = a_j(\sigma)$. The Polya–Redfield formula for a permutation group follows from (6.11) by averaging over the group.

References

1. G. Andrews, "The Theory of Partitions," Addison–Wesley, Reading, Mass., 1976.
2. N. Bourbaki, "Commutative Algebra," Chap. V, Addison–Wesley, Reading, Mass., 1972.
3. N. Bourbaki, "Groupes et algèbres de Lie," Chapitre V, Hermann, Paris, 1968.
4. N. G. de Bruijn, Polya's theory of counting, *in* "Applied Combinatorial Mathematics" E. F. Beckenbach, Ed., Chap 5, Wiley, New York, 1964.
5. W. Burnside, "Theory of Groups of Finite Order," 2nd ed., Cambridge Univ. Press, Cambridge, 1911; reprinted by Dover, New York, 1955.
6. L. Carlitz, The expansion of certain products, *Proc. Amer. Math. Soc.* 7 (1956), 558–564.
7. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Wiley–Interscience, New York, 1962.
8. E. Fischer, Über algebraische Modulsysteme und lineare homogene partielle Differentialgleichungen mit konstanten Koeffizienten, *J. Reine Angew. Math.* **140** (1911), 48–81.
9. B. Gordon, Two theorems on multipartite partitions, *J. London Math. Soc.* **38** (1963), 459–464.
10. A. Kerber, "Representations of Permutation Groups I," Springer Lecture Notes No. 240, Springer–Verlag, New York, 1971.
11. P. A. MacMahon, "Combinatory Analysis," Cambridge Univ. Press, Cambridge 1915–1916; reprinted by Chelsea, New York, 1960.
12. F. D. Murnaghan, "The Theory of Group Representations," Johns Hopkins Univ. Press, Baltimore, 1938.
13. G. Polya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen, *Acta. Math.* **68** (1937), 145–253.
14. D. P. Roselle, Coefficients associated with the expansion of certain products, *Proc. Amer. Math. Soc.* **45** (1974), 144–150.

15. I. SCHUR, Über die rationalen Darstellungen der allgemeinen linearen Gruppe, *Sitz. Preuss. Akad. Wiss. Phys. Math. Kl.* (1927), 58–75; also "Gesammelte Abhandlungen," Band III, pp. 68–85, Springer-Verlag, New York, 1973.
16. G. C. SHEPHARD AND J. A. TODD, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954), 274–304.
17. W. SPECHT, Eine Verallgemeinerung der symmetrischen Gruppe, *Schr. Math. Sem. Inst. Angew. Math. Univ. Berlin*, **1**, Heft 2 (1932), 1–32.
18. R. P. STANLEY, Theory and application of plane partitions, I, II, *Stud. Appl. Math.* **50** (1971), 167–188; 259–279.
19. R. P. STANLEY, The conjugate trace and trace of a plane partition, *J. Combinatorial Theory (A)* **14** (1973), 53–65.
20. R. STEINBERG, Differential equations invariant under finite reflection groups, *Trans. Amer. Math. Soc.* **112** (1964), 392–400.
21. K. TH. VAHLEN, Rationale Funktionen der Wurzeln; Symmetrische und Affektfunktionen, *in* "Encyklopädie der Math. Wiss.," Teil I B 3b, Teubner, Stuttgart, 1898–1904.
22. A. G. WILLIAMS, Characteristics of $G \operatorname{Wr} S_n$, *Math. Proc. Cambridge Phil. Soc.* **79** (1976), 433–441.
23. S. G. WILLIAMSON, Symmetry operators of kranz products, *J. Combinatorial Theory* **11** (1971), 122–138.
24. E. M. WRIGHT, Partitions of multi-partite numbers, *Proc. Amer. Math. Soc.* **7** (1955), 880–890.
25. O. ZARISKI AND P. SAMUEL, "Commutative Algebra," Vol. II, Van Nostrand, Princeton, N.J., 1960; reprinted by Springer-Verlag, New York, 1976.